Quiz 3

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1. We try to maximize $f(x,y,z) = x^2 + z^2 - 2xy$ under constraints $x^2 + 4y^2 + 9z^2 \leq 6$, $x \geq 0$, $y \geq 0$, and $z \geq 0$.
   (a) (1 pt) Write down the Kuhn-Tucker Lagrangian function $\tilde{L}(x,y,z,\lambda)$.
   (b) (4 pts) Suppose that the NDCQ is satisfied at the maximizer (x^*, y^*, z^*). Write down the first order conditions in the Kuhn-Tucker formulation. (You don't need to solve these equations.)

Solution:

(a) $\tilde{L}(x,y,z,\lambda) = x^2 + z^2 - 2xy - \lambda(x^2 + 4y^2 + 9z^2 - 6)$. (1 pt).

(b) There is a nonnegative λ^* such that

$$\frac{\partial \tilde{L}}{\partial x}(x^*, y^*, z^*, \lambda^*) = 2x^* - 2y^* - 2\lambda^* x^* \leq 0$$

$$\frac{\partial \tilde{L}}{\partial y}(x^*, y^*, z^*, \lambda^*) = -2x^* - 8\lambda^* y^* \leq 0$$

$$\frac{\partial \tilde{L}}{\partial z}(x^*, y^*, z^*, \lambda^*) = 2z^* - 18\lambda^* z^* \leq 0$$

$$x^* \frac{\partial \tilde{L}}{\partial x}(x^*, y^*, z^*, \lambda^*) = x^*(2x^* - 2y^* - 2\lambda^* x^*) = 0$$

$$y^* \frac{\partial \tilde{L}}{\partial y}(x^*, y^*, z^*, \lambda^*) = y^*(-2x^* - 8\lambda^* y^*) = 0$$

$$z^* \frac{\partial \tilde{L}}{\partial z}(x^*, y^*, z^*, \lambda^*) = z^*(2z^* - 18\lambda^* z^*) = 0$$

$$\frac{\partial \tilde{L}}{\partial \lambda}(x^*, y^*, z^*, \lambda^*) = -(x^* + 4y^* + 9z^* - 6) \geq 0$$

$$\lambda^* \frac{\partial \tilde{L}}{\partial \lambda}(x^*, y^*, z^*, \lambda^*) = -\lambda^*(x^* + 4y^* + 9z^* - 6) = 0$$

($\frac{1}{2}$ pt for each equation.)

2. Consider $f(x,y,z) = x^2 + z^2 - 2xy - 4xz - 2yz$ under constraints $x^2 + y^2 + z^2 = 1$ and $x + y + z = 0$. We find that $(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}})$ together with some μ_1^*, μ_2^* is a critical point of the Lagrangian function

$$L(x,y,z,\mu_1,\mu_2) = f(x,y,z) - \mu_1(x^2 + y^2 + z^2 - 1) - \mu_2(x + y + z).$$

   (a) (3 pts) Find μ_1^* and μ_2^*.
   (b) (4 pts) Write down the Hessian matrix of the Lagrangian function, $D^2_{(\vec{\mu},\vec{x})}L$, at $(x,y,z,\mu_1,\mu_2) = (\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}, \mu_1^*, \mu_2^*)$.
   (c) (3 pts) Check the second order condition and show that $f(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}})$ is a local maximum of f under the constraints.
   (d) (5 pts) Given that $f(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}})$ is the absolute maximum, estimate the maximum value of $x^2+z^2-2xy-3.9xz-2yz$ under constraints $x^2 + y^2 + z^2 = 0.98$ and $x + 1.1y + z = 0$

Solution:

(a) At $(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}})$,

$$\frac{\partial L}{\partial x} = 2x - 2y - 4z - 2\mu_1^* x - \mu_2^* = \frac{6}{\sqrt{2}} - \frac{2}{\sqrt{2}}\mu_1^* - \mu_2^* = 0$$

$$\frac{\partial L}{\partial y} = -2x - 2z - 2\mu_1^* y - \mu_2^* = -\mu_2^* = 0$$

$$\frac{\partial L}{\partial z} = 2z - 4x - 2y - 2\mu_1^* z - \mu_2^* = \frac{-6}{\sqrt{2}} + \frac{2}{\sqrt{2}}\mu_1^* - \mu_2^* = 0$$

$$\Rightarrow \mu_1^* = 3,\ \mu_2^* = 0.$$

(2 pts for listing correct equations $\frac{\partial L}{\partial x}=0$, $\frac{\partial L}{\partial y}=0$, $\frac{\partial L}{\partial z}=0$. 1 pt for solving μ_1^* and μ_2^*.)

(b) The Hessian matrix of the Lagrangian is $D^2_{(\mu_1,\mu_2,x,y,z)}L=\begin{pmatrix} 0 & 0 & -2x & -2y & -2z \\ 0 & 0 & -1 & -1 & -1 \\ -2x & -1 & 2-2\mu_1 & -2 & -4 \\ -2y & -1 & -2 & -2\mu_1 & -2 \\ -2z & -1 & -4 & -2 & 2-2\mu_1 \end{pmatrix}$

(2 pts for the right-lower 3×3 sub matrix, $D^2_{\vec{x}}L$. 1 pt for the first and second rows and columns. The answer with first and second rows and columns multiplied by -1 (the bordered Hessian matrix) is also correct.)

When $(x,y,z,\mu_1,\mu_2)=(\frac{1}{\sqrt{2}},0,\frac{-1}{\sqrt{2}},3,0)$, $D^2L=\begin{pmatrix} 0 & 0 & -\sqrt{2} & 0 & \sqrt{2} \\ 0 & 0 & -1 & -1 & -1 \\ -\sqrt{2} & -1 & -4 & -2 & -4 \\ 0 & -1 & -2 & -6 & -2 \\ \sqrt{2} & -1 & -4 & -2 & -4 \end{pmatrix}$. (1 pt)

(c) To determine whether $f(\frac{1}{\sqrt{2}},0,\frac{-1}{\sqrt{2}})$ is a local maxi or local min, we need to compute the last $(3-2)$ leading principle minors of D^2L, which is $\det D^2L$. (1 pt)
Multiply the third row with (-1) and add it to the fifth row. We obtain:

$\det D^2L=\begin{pmatrix} 0 & 0 & -\sqrt{2} & 0 & \sqrt{2} \\ 0 & 0 & -1 & -1 & -1 \\ -\sqrt{2} & -1 & -4 & -2 & -4 \\ 0 & -1 & -2 & -6 & -2 \\ 2\sqrt{2} & 0 & 0 & 0 & 0 \end{pmatrix}=-48$ which has the same sign as $(-1)^3$. (2 pts for computing the determinant.)

Hence $f(\frac{1}{\sqrt{2}},0,\frac{-1}{\sqrt{2}})$ is a local maximum.

(At $(\frac{1}{\sqrt{2}},0,\frac{-1}{\sqrt{2}})$, the gradient vectors of constraint functions are $(\sqrt{2},0,-\sqrt{2})$ and $(1,1,1)$. The vectors orthogonal to these gradient vectors are parallel to $(1,-2,1)$. And we have

$$\begin{pmatrix} a & -2a & a \end{pmatrix} D^2_{\vec{x}}L \begin{pmatrix} a \\ -2a \\ a \end{pmatrix} = \begin{pmatrix} a & -2a & a \end{pmatrix} \begin{pmatrix} -4 & -2 & -4 \\ -2 & -6 & -2 \\ -4 & -2 & -4 \end{pmatrix} \begin{pmatrix} a \\ -2a \\ a \end{pmatrix} = -24a^2<0\text{ , for all } a\neq 0,$$

which also shows that $f(\frac{1}{\sqrt{2}},0,\frac{-1}{\sqrt{2}})$ is a local maximum.)

(d) Maximize $f(x,y,z;a_1)=x^2+z^2-2xy-a_1xz-2yz$ under constraints: $h_1(x,y,z;a_2)=x^2+y^2+z^2-a_2=0$, $h_2(x,y,z;a_3)=x+a_3y+z=0$.
Suppose that the maximizer $(x^*(a_1,a_2,a_3),y^*(a_1,a_2,a_3),z^*(a_1,a_2,a_3))$ depends smoothly on parameters a_1,a_2,a_3. Then the envelope theorem says that

$$\frac{\partial}{\partial a_i}f(x^*(\vec{a}),y^*(\vec{a}),z^*(\vec{a});a_1)=\frac{\partial L}{\partial a_i}(x^*(\vec{a}),y^*(\vec{a}),z^*(\vec{a}),\mu_1^*(\vec{a}),\mu_2^*(\vec{a});a_1,a_2,a_3)\text{ for } i=1,2,3,$$

where $L(x,y,z,\mu_1,\mu_2;a_1,a_2,a_3)=f(x,y,z;a_1)-\mu_1(h_1(x,y,z;a_2))-\mu_2(h_2(x,y,z;a_3))$.
(1 pts for correctly stating the envelope theorem.)
Since $\frac{\partial L}{\partial a_1}=-xz$, $\frac{\partial L}{\partial a_2}=\mu_1$, $\frac{\partial L}{\partial a_3}=-\mu_2 y$, at $(x^*,y^*,z^*,\mu_1^*,\mu_2^*)=(\frac{1}{\sqrt{2}},0,\frac{-1}{\sqrt{2}},3,0)$, $\frac{\partial L}{\partial a_1}=\frac{1}{2}$, $\frac{\partial L}{\partial a_2}=3$, $\frac{\partial L}{\partial a_3}=0$.
(3 pts for correct partial derivatives $\frac{\partial L}{\partial a_i}, i=1,2,3$.)
We know that when $a_1=4,a_2=1,a_3=1$, the maximum value is $f(\frac{1}{\sqrt{2}},0,\frac{-1}{\sqrt{2}})=3$.
Hence when $a_1=3.9,a_2=0.98,a_3=1.1$, the maximum value is approximated by

$$f(\frac{1}{\sqrt{2}},0,\frac{-1}{\sqrt{2}})+\frac{\partial L}{\partial a_1}\times(3.9-4)+\frac{\partial L}{\partial a_2}\times(0.98-1)+\frac{\partial L}{\partial a_3}\times(1.1-1)=3+(\frac{1}{2})\times(-0.1)+3\times(-0.02)+0\times(0.1)=2.89.$$

(1 pt for the linear approximation and the final answer.)